Journal of Approximation Theory **101**, 278–288 (1999)
Article ID jath.1999.3379, available online at http://www.idealibrary.com on IDEAL®

Well Conditioned Spherical Polynomial Systems

Manfred Reimer

Fachbereich Mathematik, Universität Dortmund, D-44221 Dortmund, Germany
E-mail: reimer@math.uni-dortmund.de

Received January 10, 1998; accepted in revised form March 25, 1999

We investigate nonnegative spherical polynomials p, normalized to $p(1)=1$, which minimize the integral over the unit sphere for fixed degree. Such polynomials are useful in the construction of node systems on the sphere which support numerically stable systems of zonal functions, for instance with bounded condition numbers.

Key Words: nonnegative spherical polynomials; condition numbers; Gegenbauer polynomials.

1. INTRODUCTION AND RESULT

Let us consider linearly independent spherical functions

$$p_j : S^{r-1} \to \mathbb{R},\ j=1, ..., M,$$

S^{r-1} unit sphere in $\mathbb{R}^r$, $r \in \mathbb{N} \backslash \{1\}$, and let $\mathbb{V} := \operatorname{span}\{p_1, ..., p_M\}$. If data $y_1, ..., y_M \in \mathbb{R}$ are to be interpolated at the nodes $t_1, ..., t_M \in S^{r-1}$, which are different, then the linear system

$$\sum_{k=1}^{M} p_j(t_k)\, c_k = y_j, \qquad j=1, ..., M, \tag{1.1}$$

$c_k \in \mathbb{R}$, has to be solved, and an essential numerical requirement is that this system is as stable as possible. A similar question arises in corresponding least square approximation problems.

As a stability measure the condition number of the system (1.1) can be used, and it should be bounded even if increasingly large systems are considered. We assume that we are free in the choice of the nodes, and we assume also that the functions originate as zonal functions from a univariate function p in form

$$p_j = p(t_j \cdot\,),\ t_j \in S^{r-1}, \qquad j=1, ..., M. \tag{1.2}$$

0021-9045/99 $30.00

In this case the interpolation matrix is of the form

$$(p(t_j t_k))_{j,\,k=1,\,\ldots,\,M} \tag{1.3}$$

whose condition number depends only on the geometry of the nodes and whose diagonal is occupied by the constant values $p(1)$.

In what follows, $\mathbb{P}^r_\mu$ denotes the linear space of real r-variate polynomials with degree μ or less where $r \in \mathbb{N}$, $\mu \in \mathbb{N}_0$, while $\mathbb{P}^r_\mu(S^{r-1})$ consists of the restrictions onto S^{r-1} of the elements of $\mathbb{P}^r_\mu$. Likewise $\dot{\mathbb{P}}^r_\mu$ and $\dot{\mathbb{P}}^r_\mu(S^{r-1})$ are defined as the homogeneous polynomials with degree μ. Correspondingly $P_\mu(xy)$ and $\dot{P}_\mu(xy)$, $x, y \in S^{r-1}$, denote the reproducing kernel functions of $\mathbb{P}^r_\mu(S^{r-1})$ and $\dot{\mathbb{P}}^r_\mu(S^{r-1})$, respectively, with respect to the inner product $\langle \cdot, \cdot \rangle$ induced by the surface integral on S^{r-1}. If, in addition, $G_\mu(xy)$ denotes the reproducing kernel of the space of spherical harmonics of degree μ, then

$$P_\mu = \sum_{\kappa=0}^{\mu} G_\kappa \tag{1.4}$$

holds. For later applications we note that

$$N_\mu := \dim(\mathbb{P}^r_\mu(S^{r-1})) = \dot{N}_\mu + \dot{N}_{\mu-1} \tag{1.5}$$

is valid where

$$\dot{N}_\mu := \dim(\dot{\mathbb{P}}^r_\mu(S^{r-1})) = \binom{\mu+r-1}{r-1}. \tag{1.6}$$

For details see [7].

Naturally, we could choose $p \in \mathbb{P}^1_\mu$, while $T = \{t_1, \ldots, t_M\}$ is a fundamental system for $\mathbb{P}^r_\mu(S^{r-1})$ (i.e. $M = N_\mu$ and the corresponding evaluation functionals are linearly independent). Then we get $\mathbb{V} = \mathbb{P}^r_\mu(S^{r-1})$. A fundamental system always exists, but it is rather difficult to obtain configurations with a promising numerical behaviour, [7].

A good choice would be to let T support a *Gauß-quadrature*. Then T is necessarily fundamental. But, unfortunately, such quadrature rules do not exist apart from a few exceptional cases, [1], [2], [3]. What never fails is interpolation in *extremal* fundamental systems, [7], [8], or *hyperinterpolation* with a redundant number of evaluation functionals, [10].

Nevertheless, it is worthwhile to study the Gauß-node situation, as an example. With $p := P_\mu$ it is characterized by

$$p(t_j t_k) = \langle p_j, p_k \rangle = p(1) \cdot \delta_{j,\,k}$$

holding for $j, k = 1, ..., N_\mu$, [7], again. This means that the matrix

$$\Pi(p, M, T) := (p(t_j t_k))_{j, k = 1, ..., M}$$

is *diagonal dominant* in the strictest sense possible as

$$\text{cond}(\Pi(p, M, T)) = 1$$

(in the euclidean norm).

If a Gauß-quadrature does not exist, we can try to get a similar situation by a different choice of p, M and T. But, unfortunately, we fail to obtain diagonal dominance by this method if we insist on assuming that the number of nodes should equal the dimension N_μ. However, things change if we reduce it by some positive factor. This is the subject in what follows.

We consider the general case in which $\mathbb{W}$ is generated by the functions (1.2), but with $p \in \mathbb{P}_\mu^+$,

$$\mathbb{P}_\mu^+ := \{p \in \mathbb{P}_\mu^1 \mid p(1) = 1\,,\ p(\xi) \geqslant 0 \quad \text{for} \quad \xi \in [-1, 1)\}. \tag{1.7}$$

This is implying that the matrix $\Pi(p, M, T)$ is always nonnegative and that 1-s occur in its diagonal.

The geometry of the nodes $t_1, ..., t_M$ is defined by the assumption that the expression

$$\sum_{j=1}^{M} \sum_{k=1}^{M} p(t_j t_k), \tag{1.8}$$

which is a continuous function on $(S^{r-1})^M$, is attaining its minimum value. This implies that

$$\sum_{k \neq j} p(t_j t_k) \leqslant \sum_{k \neq j} p(x t_k)$$

holds for arbitrary $x \in S^{r-1}$ and $j \in \{1, ..., M\}$. Taking the mean value over S^{r-1} we hence obtain

$$\sum_{k \neq j} p(t_j t_k) \leqslant (M-1) \cdot I(p) \tag{1.9}$$

with the definition of

$$I(p) := \frac{1}{\omega_{r-1}} \cdot \int_{S^{r-1}} p(tx)\, d\omega(x) \tag{1.10}$$

(surface integral), which is independent of the choice of $t \in S^{r-1}$.

Therefore, in view of (1.9) and of *Gershgorin's Theorem*, a strategy to get $\operatorname{cond}(\Pi(p, M, T))$, *M fixed*, small is to assume that $I(p)$ attains its minimum value

$$m_\mu := \min\{I(p) \mid p \in \mathbb{P}_\mu^+\}, \tag{1.11}$$

while T provides the minimum value of the expression (1.8). In this case we say that T is *minimal* with degree μ and of order M.

The minimum can be calculated by using the equation

$$I(p) = \frac{\omega_{r-2}}{\omega_{r-1}} \cdot \int_{-1}^{+1} p(\xi)(1-\xi^2)^{(r-3)/2}\, d\xi. \tag{1.12}$$

By usual arguments, the right-hand side attains its minimum value in $\mathbb{P}_\mu^+$ under the condition, only, that all the μ roots of p are located in $[-1, +1)$, with even multiplicity, however, if they are located in the interior of this interval. This means that p takes the form

$$p(\xi) = \left(\frac{1+\xi}{2}\right)^\alpha \cdot P^2(\xi), \tag{1.13}$$

where $\alpha \in \{0, 1\}$ and where P is an arbitrary univariate polynomial with degree exactly $v = (\mu - \alpha)/2$ and with $P(1) = 1$.

Using this, we could evaluate the minimum by usual orthogonality arguments (Jacobi-polynomials). We prefer to use an originally multivariate method, where it suffices to know that the reproducing kernel $G(xy)$ of an arbitrary rotation-invariant polynomial space of dimension N satisfies

$$G(1) = \frac{N}{\omega_{r-1}}, \tag{1.14}$$

[7]. This knowledge enables us to identify $1/m_\mu$ directly by some space dimension:

LEMMA 1.1.

$$\frac{1}{m_\mu} = \begin{cases} N_v, & \textit{if} \quad \mu = 2v \textit{ is even,} \\ 2 \cdot \dot{N}_v, & \textit{if} \quad \mu = 2v+1 \textit{ is odd.} \end{cases}$$

The minimum value m_μ is attained if and only if

$$p(\xi) = \begin{cases} P_v^2(\xi)\, P_v^{-2}(1), & \textit{if} \quad \mu = 2v \textit{ is even,} \\ (1+\xi)/2\, \dot{P}_v^2(\xi)\, \dot{P}_v^{-2}(1) & \textit{if} \quad \mu = 2v+1 \textit{ is odd.} \end{cases}$$

Proof. First let $\mu = 2\nu$ be even, i.e., $\alpha = 0$. We can represent P in the form

$$P(\xi) = (P_\nu(\xi) + Q_\nu(\xi))/P_\nu(1),$$

where P_ν is defined as above, while Q_ν is an arbitrary univariate polynomial of degree ν and with $Q_\nu(1) = 0$. Using the reproducing property of $P_\nu(t\,\cdot)$ at $t \in S_{r-1}$ we obtain

$$P_\nu^2(1)\, Ip = \frac{1}{\omega_{r-1}} \int_{S^{r-1}} [P_\nu^2 + 2P_\nu Q_\nu + Q_\nu^2](tx)\, d\omega(x) \geqslant \frac{1}{\omega_{r-1}} P_\nu(1),$$

with equality on the right-hand side holding exactly for $Q_\nu = 0$. This yields, see (1.11), (1.14) with $G = P_\nu$,

$$m_{2\nu} = \frac{1}{N_\nu} \qquad \text{with} \quad N_\nu = \dim(\mathbb{P}_\nu). \tag{1.15}$$

Next let $\mu = 2\nu + 1$ be odd, i.e., $\alpha = 1$, and let $\dot{P}_\nu$ denote the reproducing kernel function of $\dot{\mathbb{P}}_\nu^r(S^{r-1})$, as above. Then we can represent P in the form

$$P(\xi) = (\dot{P}_\nu(\xi) + (1 - \xi)\, Q_{\nu-1}(\xi))/\dot{P}_\nu(1)$$

with $Q_{\nu-1} \in \mathbb{P}_{\nu-1}^1$. Note that

$$\dot{P}_\nu = \frac{1}{\omega_{r-1}} C_\nu^{r/2} \tag{1.16}$$

(Gegenbauer polynomial), see [7] again. Hence, using (1.10), (1.12), we get

$$\int_{S^{r-1}} (1 - (tx)^2)\, \dot{P}_\nu(tx)\, Q_{\nu-1}(tx)\, d\omega(x) = 0$$

for arbitrary $t \in S^{r-1}$. Now we use the reproducing property of $\dot{P}_\nu$ and that $\dot{P}_\nu^2$ is an even function. In view of (1.13) we then obtain

$$\begin{aligned} 2\dot{P}^2(1)\, Ip &= \frac{1}{\omega_{r-1}} \int_{S^{r-1}} [\dot{P}_\nu^2(tx) + (1 + (tx))(1 - (tx))^2)\, Q_{\nu-1}^2(tx)]\, d\omega(x) \\ &\geqslant \frac{1}{\omega_{r-1}} \dot{P}_\nu(1) \end{aligned}$$

with equality holding again if and only if $Q_\nu = 0$. Using (1.14) with $G = \dot{P}_\nu$ we hence obtain

$$m_{2\nu+1} = \frac{1}{2\dot{N}_\nu} \qquad \text{with} \quad \dot{N}_\nu = \dim(\dot{\mathbb{P}}_\nu), \tag{1.17}$$

and Lemma 1.1 is proved. Note that the dimensions occuring are explicitly known from (1.5) and (1.6).

Application of our result yields

THEOREM 1.2. *Let* $r \in \mathbb{N}\setminus\{1\}$, $\mu \in \mathbb{N}_0$. *If* p *is the polynomial as in Lemma 1.1, and if the nodes* $t_1, ..., t_M \in S^{r-1}$, $M \in \mathbb{N}$, *are furnishing the minimum value of*

$$\sum_{j=1}^{M} \sum_{k=1}^{M} p(t_j t_k) \tag{1.18}$$

with respect to the choice of $t_1, ..., t_M \in S^{r-1}$, *then*

$$\sum_{k \neq j} p(t_j t_k) \leqslant m_\mu (M-1)$$

is valid. Under the side-condition

$$M \leqslant A_\mu(q) := q \cdot \begin{cases} N_\nu, & \text{if} \quad \mu = 2\nu \text{ is even,} \\ 2\dot{N}_\nu, & \text{if} \quad \mu = 2\nu + 1 \text{ is odd,} \end{cases}$$

$0 \leqslant q < 1$, *the eigenvalues of* $\Pi(p, M, T)$ *are located in the interval* $[1-q, 1+q]$ *and the corresponding condition numbers are bounded in the form*

$$\operatorname{cond}(\Pi(p, M, T)) \leqslant \frac{1+q}{1-q}$$

independently of the polynomial degree.

Proof. Because of $p(1) = 1$ the statements can be obtained directly from (1.9) by using *Gershgorin's Theorem* together with Lemma 1.1.

2. APPLICATIONS

Obviously, if p is the polynomial of Lemma 1.1, and under the side condition of Theorem 1.2, i.e. $M = M_\mu \leqslant A_\mu(q)$, $0 \leqslant q < 1$, then the linear systems (1.1) are numerically extremely stable, and by the location of the eigenvalues it follows from [7], Theorem 13.2, that the uniform interpolation norm (Lebesgue constant) of the interpolatory operator

$$\Lambda_\mu : C(S^{r-1}) \to \operatorname{span}\{p_1, ..., p_{M_\mu}\},$$

belonging to nodes which furnish the minimum value of the expression (1.18), satifies

$$\|\Lambda_\mu\|^2 \leqslant \frac{1}{1-q} \cdot M_\mu.$$

Because of (1.5), (1.6) this means that $\|\Lambda_\mu\|$ is of order $O(\mu^{(r-1)/2})$. We are aware of the fact that the interpolation space is not the full space $\mathbb{P}^r_\mu(S^{r-1})$ (in this case the order would be the order of the minimal projection norm). In what follows we discuss least squares approximations. For the sake of simplicity we deal only with the case where $\mu = 2\nu$, $\nu \in \mathbb{N}_0$. In this case the condition number of the Gram-matrix

$$(\langle p_j, p_k \rangle)_{j,k=1,...,M_\mu},$$

is interesting. In the Gauß-case this would be unity, but in the general case it is not. Again we choose p as in Lemma 1.1, i.e., $p = P_\nu^2 / P_\nu^2(1)$. For later applications let us first recall the linearisation formulae of Rogers and Ramanujan, [5]. Using Pochhammer's symbol and the abbreviation of $C_\nu = C_\nu^\lambda$, $\lambda > 0$ fixed, they can be written in the form

$$[C_\nu]^2 = \sum_{\kappa=0}^{\nu} c_{2\kappa}^\nu C_{2\kappa}, \qquad C_\nu C_{\nu-1} = \sum_{\kappa=0}^{\nu-1} c_{2\kappa+1}^\nu C_{2\kappa+1}, \tag{2.1}$$

where the coefficients

$$c_{2\kappa}^\nu = \frac{(1)_{2\kappa}}{(2\lambda)_{2\kappa}} \cdot \frac{(\lambda)_{\nu-\kappa}}{(1)_{\nu-\kappa}} \cdot \frac{(2\lambda)_{\nu+\kappa}}{(\lambda+1)_{\nu+\kappa}} \cdot \left[\frac{(\lambda)_\kappa}{(1)_\kappa}\right]^2 \cdot \frac{2\kappa+\lambda}{\lambda}, \tag{2.2}$$

$$c_{2\kappa+1}^\nu = \frac{(1)_{2\kappa+1}}{(2\lambda)_{2\kappa+1}} \frac{(\lambda)_{\nu-\kappa-1}}{(1)_{\nu-\kappa-1}} \frac{(2\lambda)_{\nu+\kappa}}{(\lambda+1)_{\nu+\kappa}} \left[\frac{(\lambda)_\kappa(\lambda)_{\kappa+1}}{(1)_\kappa(1)_{\kappa+1}}\right] \frac{2\kappa+1+\lambda}{\lambda} \tag{2.3}$$

are positive.

It is well known that

$$P_\nu = \frac{1}{\omega_{r-1}} (C_\nu^{r/2} + C_{\nu-1}^{r/2}) \tag{2.4}$$

holds, [7], (7.20). So we obtain by using the linearisation formulae of Rogers and Ramanujan, where λ is put to be $r/2$,

$$\omega_{r-1}^2 P_\nu^2 = \sum_{\kappa=0}^{\mu} \gamma_\kappa C_\kappa^{r/2} \tag{2.5}$$

with coefficients $\gamma_\kappa > 0$. Note that

$$\gamma_{2\nu} = c_{2\nu}^{\nu}, \qquad \gamma_{2\nu-1} = 2c_{2\nu-1}^{\nu} \tag{2.6}$$

where

$$\frac{\gamma_{2\nu}}{\gamma_{2\nu-1}} = \frac{c_{2\nu}^{\nu}}{2c_{2\nu-1}^{\nu}} = \frac{\nu+\lambda-1}{2\nu+2\lambda-1} < 1 \tag{2.7}$$

holds for $\nu \in \mathbb{N}$.

In addition we can also consider the expansion

$$\omega_{r-1} P_\nu^2 = \sum_{\kappa=0}^{\mu} g_\kappa G_\kappa \tag{2.8}$$

where it is well known that

$$G_\kappa = \frac{1}{\omega_{r-1}} \left(C_\kappa^{r/2} - C_{\kappa-2}^{r/2}\right)$$

holds with $C_{-2}^{r/2} := C_{-1}^{r/2} := 0$, [7], (7.18). So we get

$$\omega_{r-1}^2 P_\nu^2 = \sum_{\kappa=0}^{\mu} (g_\kappa - g_{\kappa+2})\, C_\kappa^{r/2} \tag{2.9}$$

with $g_{\mu+2} := g_{\mu+1} := 0$, and a comparison of (2.5) and (2.9) yields

$$g_{2\kappa} = \sum_{\iota=\kappa}^{\nu} \gamma_{2\iota}, \qquad g_{2\kappa+1} = \sum_{\iota=\kappa}^{\nu} \gamma_{2\iota+1} \tag{2.10}$$

for $\kappa = 0, ..., \nu$ or $\nu - 1$, respectively. It follows

$$g_\mu < g_{\mu-2} < \cdots < g_0, \qquad g_{\mu-1} < g_{\mu-3} < \cdots < g_1. \tag{2.11}$$

Integrating (2.8) we obtain by using the kernel properties on both sides

$$g_0 = \omega_{r-1} P_\nu(1) = N_\nu = \dot{N}_\nu + \dot{N}_{\nu-1}.$$

And since $(t\cdot)$ is homogeneous of degree 1 and harmonic, we get similarly from (2.8) and (2.4), using in addition oddness/eveness and orthogonality of the functions together with (1.16),

$$\begin{aligned} g_1 &= \omega_{r-1} \int_{S^{r-1}} (tx)\, P_\nu^2(tx)\, d\omega(x) \\ &= 2\cdot \int_{S^{r-1}} \frac{1}{\omega_{r-1}}\, C_\nu^{r/2}(tx)[(tx)\, C_{\nu-1}^{r/2}(tx)]\, d\omega(x) \\ &= 2\cdot C_{\nu-1}^{r/2}(1) = 2\cdot \dot{N}_{\nu-1} < g_0, \end{aligned}$$

implying, because of (2.11),

$$\max\{g_\kappa : \kappa = 0, ..., \mu\} = g_0 = N_\nu\,. \tag{2.12}$$

The corresponding minimum can be obtained by using $g_{2\nu} = \gamma_{2\nu}$ and $g_{2\nu-1} = \gamma_{2\nu-1}$, see (2.10), which is implying $g_{2\nu} < g_{2\nu-1}$ because of (2.7). Together with (2.11) this yields

$$\min\{g_\kappa \mid \kappa = 0, 1, ..., 2\nu\} = g_{2\nu}. \tag{2.13}$$

Now let us assume that the side-condition of Theorem 1.2 holds, where $M = M_\mu$. Then, using (2.8), (2.12) and the orthogonality of different spherical harmonic spaces, we get for arbitrary $x_1, ..., x_M$

$$\begin{aligned} P_\nu^4(1) \sum_{j=1}^{M} \sum_{k=1}^{M} x_j \langle p_j, p_k \rangle x_k &= \frac{1}{\omega_{r-1}^2} \sum_{\kappa=0}^{\mu} g_\kappa^2 \sum_{j=1}^{M} \sum_{k=1}^{M} x_j G_\kappa(t_j t_k)\, x_k \\ &\leqslant \frac{1}{\omega_{r-1}^2} g_0 \sum_{\kappa=0}^{\mu} g_\kappa \sum_{j=1}^{M} \sum_{k=1}^{M} x_j G_\kappa(t_j t_k)\, x_k \\ &= \frac{1}{\omega_{r-1}} g_0 \sum_{j=1}^{M} \sum_{k=1}^{M} x_j P_\nu^2(t_j t_k)\, x_k \\ &\leqslant \frac{1}{\omega_{r-1}} g_0 P_\mu^2(1)(1+q) \sum_{j=1}^{M} x_j^2. \end{aligned}$$

Likewise we obtain with (2.13) instead of (2.12) the inequality

$$P_\nu^4(1) \sum_{j=1}^{M} \sum_{k=1}^{M} x_j < p_j, p_k > x_k \geqslant \frac{1}{\omega_{r-1}} g_{2\nu} P_\mu^2(1)(1-q) \sum_{j=1}^{M} x_j^2.$$

Together this yields

$$\operatorname{cond}((\langle p_j, p_k\rangle)_{j,k=1,\ldots,M}) \leqslant \frac{1+q}{1-q}\cdot\frac{g_0}{g_{2\nu}}$$

where

$$g_0 = N_\nu \sim \frac{2}{(r-1)!}\cdot \nu^{r-1}, \quad \text{as} \quad \nu\to\infty.$$

In the asymptotical evaluation of $g_{2\nu}$ the formula

$$\frac{(a)_\kappa}{(b)_\kappa} \sim \frac{\Gamma(b)}{\Gamma(a)}\cdot \kappa^{(a-b)} \qquad \text{as} \quad \kappa\to\infty,$$

is helpful, which holds for $a>0$, $b>0$ and which follows from Stirling's formula. By the aid of it we get from (2.2)

$$g_{2\nu} = c_{2\nu}^{\nu} \sim \frac{1}{2^{\lambda-1}\Gamma(\lambda)}\cdot \nu^{\lambda-1} \qquad \text{as} \quad \nu\to\infty$$

(with $\lambda = r/2$). Together this implies, finally,

Corollary 2.1. *Let $r\geqslant 2$, $\mu = 2\nu$, $\nu\in\mathbb{N}_0$. Under the assumptions and the side condition of Theorem* 1.2, *with $M = M_\mu$ depending on μ,*

$$\operatorname{cond}((\langle p_j, p_k\rangle)_{j,k=1,\ldots,M_\mu}) \leqslant \frac{1+q}{1-q}\cdot k_\mu$$

is valid where

$$k_\mu = \frac{g_0}{g_\mu} \sim \frac{\Gamma(r/2)}{\Gamma(r)}\cdot \mu^{r/2}$$

holds as $\mu\to\infty$.

Naturally, if the functions $p_1, \ldots, p_{M_\mu}$ would form an orthogonal system, then the condition would equal unity. But let us recall that they are nonnegative, and hence never orthogonal. But even if we forget this, there cannot exist an orthogonal system of *zonal functions* for $M_\mu = N_\mu$, as Gauß-quadratures do not exist, in general. So the statement of Corollary 2.1 is nontrivial, as the condition is, roughtly speaking, increasing at the order of the square root of the rank of the matrix, at least for large r.

Because of (2.8) and by the reproducing property of the $G_\kappa - s$ the matrix $\Pi(p, M, T)$ is a positive linear combination of the Gram-matrices

$$(G_\kappa(t_j t_k))_{j,k=1,..,M} = (\langle G_\kappa(t_j \cdot), G_\kappa(t_k \cdot)\rangle)_{j,k=1,..,M}$$

and hence *positive semidefinite*. If M exceeds the value of N_μ, it is necessarily *singular*. But apart from this the result of Theorem 1.2 can be compared with results on *positive definite functions* in the sense of [9], [4]. For more recent work on this topic we refer to [6].

REFERENCES

1. E. Bannai and R. M. Damerell, Tight spherical designs I, *J. Math. Soc. Japan* **31** (1979), 199–207.
2. E. Bannai and R. M. Damerell, Tight spherical designs II, *J. Lond. Math. Soc.* **21**(2) (1980), 13–30.
3. L. Bos, Some remarks on the Fejér problem for Lagrange interpolation in several variables, *J. Approx. Theory* **60** (1990), 133–140.
4. E. W. Cheney, Approximation using positive definite functions, *in* "Approximation Theory VIII" (C. Chui and L. L. Schumaker, Eds.), Vol. 1, Approximation and Interpolation, pp. 145–168, World Scientific, Singapore, 1995.
5. G. Gasper, Rogers' linearization formula for the continuous q-ultraspherical polynomials and quadratic transformation formulas, *SIAM J. Math. Anal.* **16** (1985), 1061–1071.
6. K. Jetter, J. Stöckler and J. D. Ward, Error estimates for scattered data interpolation on spheres, preprint, 1997.
7. M. Reimer, "Constructive Theory of Multivariate Functions," BI-Wissenschaftsverlag, Mannheim, 1990.
8. M. Reimer, B. Sündermann, A Remez-type algorithm for the calculation of extremal fundamental systems for polynomial spaces over the sphere, *Computing* **37** (1986), 43–58.
9. I. J. Schoenberg, Positive definite functions on spheres, *Duke Math. J.* **9** (1942), 96–108.
10. I. H. Sloan, Interpolation and hyperinterpolation on the sphere, *in* "Multivariate Approximation" (W. Haußmann, K. Jetter, and M. Reimer, Eds.), pp. 255–268, Akademie Verlag, Berlin, 1997.